

Galois correspondence theorem for Picard-Vessiot extensions

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Abstract

In this paper, we generalize the definition of the differential Galois group and the Galois correspondence theorem established previously for Picard-Vessiot extensions of real differential fields with real closed field of constants to any Picard-Vessiot extension.

1 Introduction

For a homogeneous linear differential equation $\mathcal{L}(Y) = 0$ defined over a differential field K with field of constants C , a Picard-Vessiot extension is a differential field L , differentially generated over K by a fundamental system of solutions of $\mathcal{L}(Y) = 0$ and with constant field equal to C . A classical result states that the Picard-Vessiot extension exists and is unique up to K -differential isomorphism in the case C algebraically closed (see [5]). Recently, an existence and uniqueness result for Picard-Vessiot extensions has been established in the case when the differential field K is a formally real field (resp. a formally p -adic field) with real closed (resp. p -adically closed of the same rank than K) field of constants C ([1]). In [2] we presented a Galois correspondence theorem for Picard-Vessiot extensions of formally real differential fields with real closed field of constants. In this paper we establish a Galois correspondence theorem for general Picard-Vessiot extensions, valid in particular for Picard-Vessiot extensions of formally p -adic differential fields with p -adically closed field of constants.

Kolchin introduced the concept of strongly normal differential field extension and obtained a satisfactory Galois correspondence theorem for this class of extensions without assuming the field of constants of the differential base field to be algebraically closed (see [4] Chapter VI). Note that, for a

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strongly normal extension $L|K$, in the case when the constant field of K is not algebraically closed, the differential Galois group is no longer the group $DAut_K L$ of K -differential automorphisms of L , rather one has to consider as well K -differential morphisms of L in larger differential fields. It is worth noting that a Picard-Vessiot extension is always strongly normal. In the case of Picard-Vessiot extensions, we can adopt a definition of the differential Galois group inspired by Kolchin's but simpler than his one. We obtain then a Galois correspondence theorem which classifies intermediate differential fields of a Picard-Vessiot extension in terms of its differential Galois group.

In this paper, we shall deal with fields of characteristic 0. For the sake of simplicity in the exposition we consider ordinary differential fields.

2 Main result

We recall now the precise definition of Picard-Vessiot extension.

Definition 1. Given a homogeneous linear differential equation

$$\mathcal{L}(Y) := Y^{(n)} + a_{n-1}Y^{(n-1)} + \dots + a_1Y' + a_0Y = 0$$

of order n over a differential field K , a differential extension $L|K$ is a *Picard-Vessiot extension* for \mathcal{L} if

1. $L = K\langle\eta_1, \dots, \eta_n\rangle$, where η_1, \dots, η_n is a fundamental set of solutions of $\mathcal{L}(Y) = 0$ in L .
2. Every constant of L lies in K , i.e. $C_K = C_L$.

As mentioned in the introduction, a Picard-Vessiot extension is strongly normal. Hence, the fundamental theorem established by Kolchin in [4] chapter VI applies to Picard-Vessiot extensions. However, for a strongly normal extension $L|K$, Kolchin defines the differential Galois group $DGal(L|K)$ by means of differential K -isomorphisms of L in the differential universal extension U of L . He obtains then that $DGal(L|K)$ has the structure of an algebraic group defined over the field of constants C_U of U . Afterwards, by using the notion of specialization in Weil's algebraic geometry, he proves that there exists an algebraic group G defined over the field of constants C of K such that $G(C_U) = DGal(L|K)$. In this section, we give a more direct definition of the differential Galois group of a Picard-Vessiot extension, we

endow it with a linear algebraic group structure over C and establish a Galois correspondence theorem in our setting.

2.1 Differential Galois group

Let K be a differential field with field of constants C , let \mathcal{K} be a differential closure of K . Let \overline{C} denote an algebraic closure of C contained in \mathcal{K} and let $\{\alpha_i\}_{i \in I}$ be a C -basis of \overline{C} . In the sequel, differential field extensions of K will be assumed to be subfields of \mathcal{K} . For any such extension F (included $F = K$), we shall denote by \overline{F} the composition field of F and \overline{C} inside \mathcal{K} . If the field of constants of F is equal to C , the extensions $\overline{C}|C$ and $F|C$ are linearly disjoint and then $\{\alpha_i\}_{i \in I}$ is an F -basis of \overline{F} . For a Picard-Vessiot extension $L|K$, we shall consider the set $DHom_K(L, \overline{L})$ of K -differential morphisms from L into \overline{L} . We shall see that we can define a group structure on this set and we shall take it as the differential Galois group $\mathrm{DGal}(L|K)$ of the Picard-Vessiot extension $L|K$. We shall prove that it is a C -defined (Zariski) closed subgroup of some \overline{C} -linear algebraic group.

We observe that we can define mutually inverse bijections

$$\begin{array}{ccc} DHom_K(L, \overline{L}) & \rightarrow & DAut_{\overline{K}\overline{L}} \\ \sigma & \mapsto & \widehat{\sigma} \end{array}, \quad \begin{array}{ccc} DAut_{\overline{K}\overline{L}} & \rightarrow & DHom_K(L, \overline{L}) \\ \tau & \mapsto & \tau|_L \end{array},$$

where $\widehat{\sigma}$ is the extension of σ to \overline{L} . For an element $\sum \lambda_i \alpha_i$ in \overline{L} , where $\lambda_i \in L$, we define $\widehat{\sigma}(\sum \lambda_i \alpha_i) = \sum \sigma(\lambda_i) \alpha_i$. We may then transfer the group structure from $DAut_{\overline{K}\overline{L}}$ to $DHom_K(L, \overline{L})$. Let us note that $DAut_{\overline{K}\overline{L}}$ is the differential Galois group of the Picard-Vessiot extension $\overline{L}|\overline{K}$.

Let now η_1, \dots, η_n be C -linearly independent elements in L such that $L = K\langle \eta_1, \dots, \eta_n \rangle$ and $\sigma \in DHom_K(L, \overline{L})$. We have then $\sigma(\eta_j) = \sum_{i=1}^n c_{ij} \eta_i$, $1 \leq j \leq n$, with $c_{ij} \in \overline{C}$. We may then associate to σ the matrix (c_{ij}) in $\mathrm{GL}(n, \overline{C})$. The proofs of Propositions 16 and 17 and Corollary 18 in [2] remain valid in our present setting. We obtain then the following results.

Proposition 2. *Let K be a differential field with field of constants C , $L = K\langle \eta_1, \dots, \eta_n \rangle$ a Picard-Vessiot extension of K , where η_1, \dots, η_n are C -linearly independent. There exists a set S of polynomials $P(X_{ij})$, $1 \leq i, j \leq n$, with coefficients in C such that*

- 1) *If $\sigma \in DHom_K(L, \overline{L})$ and $\sigma(\eta_j) = \sum_{i=1}^n c_{ij} \eta_i$, then $P(c_{ij}) = 0, \forall P \in S$.*

- 2) Given a matrix $(c_{ij}) \in \text{GL}(n, \overline{C})$ with $P(c_{ij}) = 0, \forall P \in S$, there exists a differential K -morphism σ from L to \overline{L} such that $\sigma(\eta_j) = \sum_{i=1}^n c_{ij} \eta_i$.

The preceding proposition gives that $\text{DGal}(L|K)$ is a C -defined closed subgroup of $\text{GL}(n, \overline{C})$.

Proposition 3. *Let K be a differential field with field of constants C , $L|K$ a Picard-Vessiot extension. For $a \in L \setminus K$, there exists a K -differential morphism $\sigma : L \rightarrow \overline{L}$ such that $\sigma(a) \neq a$.*

For a subset S of $\text{DGal}(L|K)$, we set $L^S := \{a \in L : \sigma(a) = a, \forall \sigma \in S\}$.

Corollary 4. *Let K be a differential field with field of constants C , $L|K$ a Picard-Vessiot extension. We have $L^{\text{DGal}(L|K)} = K$.*

2.2 Fundamental theorem

Let K be a differential field with field of constants C and $L|K$ a Picard-Vessiot extension. For a closed subgroup H of $\text{DGal}(L|K)$, L^H is a differential subfield of L containing K . If E is an intermediate differential field, i.e. $K \subset E \subset L$, then $L|E$ is a Picard-Vessiot extension and $\text{DGal}(L|E)$ is a C -defined closed subgroup of $\text{DGal}(L|K)$.

Theorem 5. *Let $L|K$ be a Picard-Vessiot extension, $\text{DGal}(L|K)$ its differential Galois group.*

1. *The correspondences*

$$H \mapsto L^H \quad , \quad E \mapsto \text{DGal}(L|E)$$

define inclusion inverting mutually inverse bijective maps between the set of C -defined closed subgroups H of $\text{DGal}(L|K)$ and the set of differential fields E with $K \subset E \subset L$.

2. *The intermediate differential field E is a Picard-Vessiot extension of K if and only if the subgroup $\text{DGal}(L|E)$ is normal in $\text{DGal}(L|K)$. In this case, the restriction morphism*

$$\begin{array}{ccc} \text{DGal}(L|K) & \rightarrow & \text{DGal}(L|E) \\ \sigma & \mapsto & \sigma|_E \end{array}$$

induces an isomorphism

$$\mathrm{DGal}(L|K)/\mathrm{DGal}(L|E) \simeq \mathrm{DGal}(E|K).$$

Proof. 1. It is clear that both maps invert inclusion. If E is an intermediate differential field of $L|K$, we have $L^{\mathrm{DGal}(L|E)} = E$, taking into account that $L|E$ is Picard-Vessiot and corollary 4. For H a C -defined closed subgroup of $\mathrm{DGal}(L|K)$, the equality $H = \mathrm{DGal}(L|L^H)$ follows from the correspondent equality in Picard-Vessiot theory for differential fields with algebraically closed field of constants (see e.g. [3] theorem 6.3.8).

2. If E is a Picard-Vessiot extension of K , then \overline{E} is a Picard-Vessiot extension of \overline{K} and so $\mathrm{DGal}(L|E)$ is normal in $\mathrm{DGal}(L|K)$. Reciprocally, if $\mathrm{DGal}(L|E)$ is normal in $\mathrm{DGal}(L|K)$, then the subfield of \overline{L} fixed by $\mathrm{DGal}(L|E)$ is a Picard-Vessiot extension of \overline{K} . Now, this field is \overline{E} . So, \overline{E} is differentially generated over \overline{K} by a \overline{C} -vector space V of finite dimension. Let $\{v_1, \dots, v_n\}$ be a \overline{C} -basis of V . We may write each $v_j, 1 \leq j \leq n$ as a linear combination of the elements α_i with coefficients in E . Now, there is a finite number of α_i 's appearing effectively in these linear combinations. Let \tilde{C} be a finite Galois extension of C containing all these α_i 's. We have then $v_i \in \tilde{E} := \tilde{C} \cdot E$ and \tilde{E} is differentially generated over $\tilde{K} := \tilde{C} \cdot K$ by V . We may extend the action of $\mathrm{Gal}(\tilde{C}|C)$ to \tilde{E} and consider the translate $c(V)$ of V by $c \in \mathrm{Gal}(\tilde{C}|C)$. Let $\tilde{V} = \bigoplus_{c \in \mathrm{Gal}(\tilde{C}|C)} c(V)$. We have that \tilde{E} is differentially generated over \tilde{K} by \tilde{V} and \tilde{V} is $\mathrm{Gal}(\tilde{C}|C)$ -stable, hence E is differentially generated over K by the C -vector space $\tilde{V}^{\mathrm{Gal}(\tilde{C}|C)} = \{y \in V : c(y) = y, \forall c \in \mathrm{Gal}(\tilde{C}|C)\}$. Hence $E|K$ is a Picard-Vessiot extension. The last statement of the theorem follows from the fundamental theorem of Picard-Vessiot theory in the case of algebraically closed fields of constants ([3] theorem 6.3.8). \square

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